Name: Feedback

SA402 · Dynamic and Stochastic Models — Fall 2022 · Uhan

Exam 2 – 11/9/2022

Instructions

- You have 50 minutes to complete this exam.
- You may use your plebe-issue TI-36X Pro calculator.
- You may not use any other materials.
- **No collaboration allowed.** All work must be your own.
- **Show all your work.** To receive full credit, your solutions must be completely correct, sufficiently justified, and easy to follow.
- Keep this booklet intact.
- **Do not discuss the contents of this exam with any midshipmen until it is returned to you.**

Problem	Weight	Score
1a	1.25	
1b	1.25	
1c	1.25	
2	1.25	
3a	1.25	
3b	1.25	
3c	1.25	
4	1.25	
Total		/ 100

Problem 0. Copy and sign the honor statement below. This exam will not be graded without a signed honor statement.

> The Naval Service I am a part of is bound by honor and integrity. I will not compromise our values by giving or receiving unauthorized help on this exam.

Problem 1. Lemon is a scooter sharing service that operates in Simplexville. They have hired you to study the movement of its scooters between three regions in Simplexville: Downtown, North, and South. The company's data science team has modeled the movement of a scooter as a Markov chain with 3 states. The states $1, 2, 3$ correspond to Downtown, North, and South, respectively, and each time step corresponds to one scooter trip. When a scooter reaches its destination, it stays in the destination region until it is used again. The one-step transition probability matrix for a scooter is:

$$\mathbf{P} = \begin{bmatrix} 0.65 & 0.25 & 0.10 \\ 0.30 & 0.50 & 0.20 \\ 0.35 & 0.20 & 0.45 \end{bmatrix}$$

a. Suppose at the beginning of the day, 50% of the bikes are in the Downtown region, 25% in the North region, and 25% in the South region. What is the probability that a bike randomly chosen at the beginning of the day will be in the Downtown region after 5 trips?

See Example 3 in Lesson 8, Problems 1b, 2c, and 3d from the Lesson 8 Exercises, and Problem 2a from the Exam 2 Review Problems for similar examples.

b. What is the probability that a scooter starting in the South region will be in the Downtown region after 3 trips?

See Example 2 in Lesson 8 and Problems 2b and 3c from the Lesson 8 Exercises for similar examples.

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Here is the transition probability matrix for Problem 1 again:

$$\mathbf{P} = \begin{bmatrix} 0.65 & 0.25 & 0.10 \\ 0.30 & 0.50 & 0.20 \\ 0.35 & 0.20 & 0.45 \end{bmatrix}$$

c. What is the probability that a bike starts in the Downtown region, stays in either the Downtown or South regions for 6 trips, and then goes to the North region in the 7th trip?

See Example 4 in Lesson 8, and Problems 1c, 2d, and 3e from the Lesson 8 Exercises for similar examples.

Problem 2. You have just been hired as an analyst in the Cauchy County Department of Health and Human Services. Your predecessor developed a model of the county population, in which each citizen can be classified as living in one of three location types: urban, rural, or suburban. In their model, the state of the system is defined as a citizen's current location type, and the time index is defined to be 1 year.

Describe what assumptions need to be made in order for the time-stationarity property to hold. (You do not need to discuss whether these assumptions are realistic.)

See Lesson 10 for the definition of the time-stationarity property for Markov chains.

Many of you wrote that the time-stationarity property requires that the probability that a citizen lives in an urban, rural, or suburban location be the same from year to year. Be careful. Note that the time-stationarity property is a statement about the one-step transition probabilities of a Markov chain. What does a transition between states correspond to in this model?

Problem 3. An autonomous UUV has been programmed to move randomly between 6 regions according to a Markov chain. Looking at the documentation written by the programmer, you find the following one-step transition matrix:

$$\mathbf{P} = \begin{bmatrix} 0.10 & 0.25 & 0.15 & 0.10 & 0.05 & 0.35 \\ 0.20 & 0.10 & 0.40 & 0.10 & 0.15 & 0.05 \\ 0 & 0 & 0.25 & 0.75 & 0 & 0 \\ 0 & 0 & 0.65 & 0.35 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.70 & 0.30 \\ 0 & 0 & 0 & 0 & 0.50 & 0.50 \end{bmatrix}$$

a. Do regions 1 and 2 form an irreducible set of states? Why or why not?

See the top of page 3 in Lesson 9 for the definition of an irreducible set of states. Also see Problem 1a in the Lesson 9 Exercises and Problem 2b in the Exam 2 Review Problems for similar examples.

b. Suppose the UUV reaches region 4. What is the long-run fraction of time it spends in region 4?

See Example 2 in Lesson 9, Problem 1c in the Lesson 9 Exercises, and Problem 2c in the Exam 2 Review Problems for similar examples.

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Here is the transition probability matrix for Problem 3 again:

$$\mathbf{P} = \begin{bmatrix} 0.10 & 0.25 & 0.15 & 0.10 & 0.05 & 0.35 \\ 0.20 & 0.10 & 0.40 & 0.10 & 0.15 & 0.05 \\ 0 & 0 & 0.25 & 0.75 & 0 & 0 \\ 0 & 0 & 0.65 & 0.35 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.70 & 0.30 \\ 0 & 0 & 0 & 0 & 0.50 & 0.50 \end{bmatrix}$$

c. Suppose the UUV starts in region 2. What is the probability that the UUV eventually ends up in region 5 or 6?

See Problem 1d from the Lesson 9 Exercises for a similar example.

Problem 4. You have been put in charge of inventory management at the Poisson Fish Market. The inventory system for tuna works as follows:

1. Observe the number of crates of tuna in inventory at the beginning of the day. Call this number n. The storage area for tuna can hold at most 4 crates.
2. If $n \in \{0, 1\}$, then order $4 - n$ crates. If $n \in \{2, 3, 4\}$, then order 0 crates. Orders are delivered immediately.
3. Throughout the day, some of these crates of tuna are sold. With probability 1/3, 0 crates are sold. With probability 1/2, 1 crate is sold. With probability 1/6, 2 crates are sold.
4. Observe the number of crates in inventory at the beginning of the next day.

When you start observing the system, there are 2 crates of tuna in inventory.

Model this setting as a Markov chain by defining:

- the state space,
- the meaning of one time step in the setting's context,
- the meaning of the state visited in the nth time step in the setting's context, and
- the one-step transition probabilities and initial state probabilities.

Most of you were on the right track with this problem, but had a critical error or were missing some parts.

- You were asked to define the initial state probabilities. Don't forget these!
- Note that ordering crates at the beginning of the day does not prevent crates being sold throughout the day. For example, if there is 1 crate in inventory at the beginning of the day, then 3 crates are ordered and delivered immediately, meaning there are 4 crates of tuna available to be sold throughout the day according to the probabilities described in Step 3.
- Be careful about the meaning of one time step: when do you observe the number of crates in inventory?